Journal of Statistical Physics, Vol. 115, Nos. 5/6, June 2004 (© 2004)

Pure Point Spectrum for Two-Level Systems in a Strong Quasi-Periodic Field

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Received September 22, 2003; accepted December 29, 2003

We consider two-level atoms in a strong external quasi-periodic field with Diophantine frequency vector. We show that if the field is an analytic function with zero average, then for a large set of values of its frequency vector, characterized by imposing infinitely many Diophantine conditions, the spectrum of the quasi-energy operator is pure point, as in the case of nonzero average which was already known in literature.

KEY WORDS: Two-level systems; pure point spectrum; generalized Riccati equation; small divisors; quasi-periodic solutions; trees; multiscale analysis; resummation of divergent series; Cantor set.

1. INTRODUCTION

Consider a two-level system in a quasi-periodic external field.[10] The corresponding Hamiltonian is given by

$$H = \varepsilon\sigma_3 - f(t)\,\sigma_1, \tag{1.1}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices and $f(t)$ is assumed to be a real analytic quasi-periodic function with frequency (or rotation) vector $\boldsymbol{\omega} \in \mathbb{R}^d$; the real parameter ε measures half the spacing between the unperturbed energy levels.

If we write

$$f(t) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} f_{\boldsymbol{\nu}}, \qquad \overline{f_{\boldsymbol{\nu}}} = f_{-\boldsymbol{\nu}}, \tag{1.2}$$

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with the bar denoting complex conjugation, then the Hamiltonian (1.1) can be seen as a function of $\boldsymbol{\theta} = \boldsymbol{\omega} t$; we shall write $H = H(\boldsymbol{\omega} t)$.

The model has been recently considered in refs. 5 and 11, where the spectrum for the quasi-energy operator was studied. The latter is defined on the extended Hilbert space $\mathscr{H} = \mathbb{C}^2 \times L^2(\mathbb{T}^d, \mu)$, with $\mu = 1/(2\pi)^d \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_d$, as

$$K = -i\boldsymbol{\omega} \cdot \frac{\partial}{\partial \boldsymbol{\theta}} + H(\boldsymbol{\theta}); \tag{1.3}$$

see refs. 2 and 9.

References 5 and 11 explicitly deal with the case $d = 2$, in the regimes, respectively, of large ε (small external field) and small ε (large external field): in both papers the spectrum of the quasi-energy operator is shown to be pure point for $\alpha = \omega_1/\omega_2$ Diophantine and excluding a further small set of resonant values. More precisley, ref. 11 shows that one can reduce the case of small ε to the case of large ε solved in ref. 5, provided that the average f_0 of the external field is nonvanishing: this is accomplished by performing a unitary transformation which casts the quasi-energy operator into the same form as in the case of large ε, but one needs f_0 to be not zero. Note that external fields with vanishing average represent sort of a degenerate situation, as in such a case the levels of the reference free system, obtained through the aforementioned unitary transformation, have the same energy for $\varepsilon = 0$. So the case of ε small and $f_0 = 0$ is still left as an open problem in literature.

The time-dependent Schrödinger equation for the Hamiltonian (1.1) is given by

$$i \frac{\partial}{\partial t} \psi(t) = H(\boldsymbol{\omega} t) \, \psi(t). \tag{1.4}$$

In ref. 2 the solutions of Eq. (1.4) are shown to be expressible in terms of any particular solution g of the generalized Riccati equation

$$\frac{\mathrm{d}G}{\mathrm{d}t} - iG^2 - 2if(t) \, G + i\varepsilon^2 = 0. \tag{1.5}$$

In particular in ref. 2 it was found that quasi-periodic solutions of the generalized Riccati equation exist in the form of formal power series in ε, but such series were argued to be in general divergent. In ref. 8 it was proved that for small ε quasi-periodic solutions exist indeed, at least for values of ε belonging to a Cantor set centered around the origin.

To come back to the original problem about the spectrum of the quasi-energy operator is not so immediate, as one has to check some properties of the solution of the generalized Riccati equation, which are not obvious (see Section 7 in ref. 2; we shall come back to this in Section 3).

Furthermore the problem studied in refs. 2 and 8 differs from that considered in refs. 5 and 11, as one fixes the frequencies $\omega_1,\dots,\omega_d$, with $d \geqslant 1$, and, by imposing a Diophantine condition on the rotation vector $(\omega_1,\dots,\omega_d, f_0)$ if $f_0 \neq 0$ and on the rotation vector $(\omega_1,\dots,\omega_d)$ if $f_0 = 0$, one finds quasi-periodic solutions by requesting further conditions on the parameter ε. But we can also consider the same problem as in refs. 5 and 11: for fixed ε we find conditions on the rotation vectors, to be added to the usual Diophantine one, in order to have quasi-periodic solutions.

The conclusions are summarized in the following result. For simplicity we state the result only when one has $f_0 = 0$, as this is the case which will be considered later, but the proof immediately extends to any analytic function $f(t)$ in (1.5), without requesting any condition on its average. We assume also the same nondegeneracy condition of the external field as in ref. 8 (which corresponds to the condition of case (1) of Theorem 2.2 in ref. 2), but very likely such a condition can be easily eliminated.

Theorem 1. Consider the generalized Riccati equation (1.5), with f a real analytic quasi-periodic function of the form (1.2), with $f_0 = 0$, and let $\mathcal{D}_\tau(C_0)$ be the set of vectors $\boldsymbol{\omega} \in \mathbb{R}^d$ satisfying the Diophantine condition

$$|\boldsymbol{\omega} \cdot \mathbf{v}| > C_0 \, |\mathbf{v}|^{-\tau} \qquad \forall \mathbf{v} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}, \tag{1.6}$$

with Diophantine constants $C_0 > 0$ and $\tau > d-1$. Assume that, by defining

$$Q(t) = \exp\left(2i \int_0^t \mathrm{d}t' \, f(t')\right) = \sum_{\mathbf{v} \in \mathbb{Z}^d} e^{i\mathbf{v}\cdot\boldsymbol{\omega}t} Q_{\mathbf{v}}, \tag{1.7}$$

one has $Q_{\mathbf{0}} \neq 0$. There exists a positive constant ε_0 such that, for all $|\varepsilon| < \varepsilon_0$ and for any ball $V \subset \mathbb{R}^d$ of Lebesgue measure 1, there are a positive constant C_0, a positive constant b_V and a set $\Omega_\varepsilon(V) \subset V \cap \mathcal{D}_\tau(C_0)$ of relative Lebesgue measure

$$\operatorname{meas}(\Omega_\varepsilon(V)) \geqslant 1 - b_V C_0, \tag{1.8}$$

such that for all $\boldsymbol{\omega} \in \Omega_\varepsilon(V)$ the generalized Riccati equation (1.5) admits a particular solution of the form

$$g(t) \equiv \bar{g}(t; \varepsilon, \boldsymbol{\omega}) = \tilde{g}(\boldsymbol{\omega}t; \varepsilon, \boldsymbol{\omega}), \tag{1.9}$$

with $\tilde{g}(\boldsymbol{\psi}; \varepsilon, \boldsymbol{\omega})$ analytic and 2π-periodic in $\boldsymbol{\psi}$.

Note that the condition $f_0 = 0$ is used to expand the function $Q(t)$ as in (1.7): for $f_0 \neq 0$, the rotation vector would be $(\omega_1,..., \omega_d, f_0)$; see refs. 2 and 8.

In the particular case in which one has $f_0 = 0$ (which is the case left out in literature), we show in Section 3 that it is possible to obtain informations about the spectrum of the corresponding quasi-energy operator: we find that also in such a case, for a set of values $\boldsymbol{\omega} = (\omega_1,..., \omega_d)$ of large relative measure, the spectrum is pure point. More precisely the following statement can be made (covering also the cases known from the literature).

Theorem 2. Consider the quasi-energy operator K in (1.3), with $H = H(\boldsymbol{\theta})$ given by (1.1) and f a real analytic quasi-periodic function of the form (1.2). Let $\mathcal{D}_\tau(C_0)$ be defined as in Theorem 1. Assume that, by defining $Q(t)$ as in (1.7), one has $Q_0 \neq 0$. There exists a positive constant ε_0 such that, for all $|\varepsilon| < \varepsilon_0$ and for any ball $V \subset \mathbb{R}^d$ of Lebesgue measure 1, there are a positive constant C_0, a positive constant b_V and a set $\Omega_\varepsilon(V) \subset V \cap \mathcal{D}_\tau(C_0)$ of relative Lebesgue measure

$$\operatorname{meas}(\Omega_\varepsilon(V)) \geqslant 1 - b_V C_0, \tag{1.10}$$

such that for all $\boldsymbol{\omega} \in \Omega_\varepsilon(V)$ the spectrum of K is pure point.

For $f_0 \neq 0$ the proof can be found in ref. 11. For $f_0 = 0$ the proof is carried out in Section 3; the set $\Omega_\varepsilon(V)$ and the constant b_V can be taken the same as in Theorem 1.

Of course it would be interesting also to investigate systems with infinite (possibly degenerate) levels in a quasi-periodic field. Partial results in this direction have been obtained, for instance, in refs. 6 and 7, where the case of weak periodic field has been studied, and in ref. 1, where quasi-periodic fields has been considered for systems with infinetely many non-degenerate eigenvalues growing fast enough. We note that the technical difficulties arising in studying such problems are strongly related to those characterizing the problem of finding periodic and quasi-periodic solutions for Hamiltonian systems: the case of a finite number of levels is similar to the case of elliptic lower-dimensional tori for finite-dimensional Hamiltonian systems (and the case of two levels is analogous to the case in which there is only one normal frequency), while the cases of systems with infinitely many levels in a periodic or quasi-periodic field correspond, respectively, to the cases of periodic or quasi-periodic solutions in Hamiltonian PDE systems. In particular it is well known that, for the latter systems, the problem of proving existence of quasi-periodic solutions is more difficult than that of proving existence of periodic solutions. By pursuing further such an analogy the case of zero-average external field we consider in this paper corresponds to the case of a system with degenerate free Hamiltonian, and this requires perturbation theory in presence of degeneracies, as already pointed out in ref. 11.

2. RESULTS ABOUT THE SOLUTIONS OF THE GENERALIZED RICCATI EQUATION

To prove Theorem 1 we proceed as in ref. 8, where we studied the case in which $\boldsymbol{\omega} = (\omega_1,..., \omega_d)$ is fixed and ε is a parameter to be varied. As in ref. 8 we write the solution of the generalized Riccati equation (1.5) as $G = i\varepsilon Q u$, where

$$u = u(t; \varepsilon, \boldsymbol{\omega}) = \sum_{k=0}^{\infty} \varepsilon^k \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} e^{i\boldsymbol{\omega} \cdot \boldsymbol{\nu} t} u_{\boldsymbol{\nu}}^{[k]}(\varepsilon, \boldsymbol{\omega}) \tag{2.1}$$

admits the renormalized tree expansion envisaged in Section 6 in ref. 8: we refer there for notations and details. In fact note that the tree expansion and the multiscale decomposition are carried out at fixed $\boldsymbol{\omega}$ and ε, so that it is not important if either we are supposing that $\boldsymbol{\omega}$ is fixed and ε varies or vice versa.

As we want to consider $\boldsymbol{\omega}$ as a parameter to be varied we shall write $g^{[n]}(\boldsymbol{\omega}, \boldsymbol{\nu}; \varepsilon)$ and $M^{[n]}(\boldsymbol{\omega}, \boldsymbol{\nu}; \varepsilon)$ instead of $g^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon)$ and $M^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon)$ as in ref. 8; in the same way we shall write $\mathscr{M}^{[n]}(\boldsymbol{\omega}, \boldsymbol{\nu}; \varepsilon)$.

The condition $Q_0 \neq 0$ is imposed in order to show, by a first order analysis, that the coefficients $u_{\boldsymbol{\nu}}^{[k]}$ are formally well defined; again we refer to ref. 8 (and ref. 2) for details.

The only real difference with respect to ref. 8, in the renormalized expansion, is in the Diophantine conditions. For fixed ε we assume that one has

$$|i\boldsymbol{\omega} \cdot \boldsymbol{\nu} - \mathscr{M}^{[n]}(\boldsymbol{\omega}, \boldsymbol{\nu}; \varepsilon)| \geqslant \frac{C_0}{2^{(n+1)/2} |\nu|^\tau} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^d \setminus \{\mathbf{0}\} \quad \text{and} \quad \forall n \geqslant -1, \tag{2.2}$$

with the same Diophantine constants C_0 and τ as in (1.6). We call Ω_ε the set of $\boldsymbol{\omega}$ for which the Diophantine conditions (2.2) are satisfied; for any subset $V \subset \mathbb{R}^d$ we denote with $\Omega_\varepsilon(V)$ the set $\Omega_\varepsilon \cap V$.

Note that the set Ω_ε is contained inside the set $\mathscr{D}_\tau(C_0)$ of Diophantine vectors with Diophantine constants C_0 and τ, which corresponds to (2.2) for $n = -1$ (when $\mathscr{M}^{[-1]}(\boldsymbol{\omega}, \boldsymbol{\nu}; \varepsilon)$ is identically vanishing, see ref. 8).

Suppose that ε_1 is such that the series

$$\sum_{k=0}^{\infty} \varepsilon^k \sum_{\mathbf{v} \in \mathbb{Z}^d} e^{i\boldsymbol{\omega}\cdot\mathbf{v}t} \bar{u}_{\mathbf{v}}^{(k)}, \qquad \bar{u}_{\mathbf{v}}^{(k)} = \sum_{\theta \in \Theta_{k,\mathbf{v}}^{\mathcal{R}}} \overline{\mathrm{Val}}(\theta), \tag{2.3}$$

obtained by replacing the propagators $g^{[n_\ell]}$ with $G 2^{n_\ell+1} C_0^{-1}$ (for some constant G), converges for $|\varepsilon| \leqslant \varepsilon_1$. In the following discussion we shall consider values of ε such that $|\varepsilon| < \varepsilon_0$, with $\varepsilon_0 \in (0, \varepsilon_1]$ small enough (how small can be deduced from the discussion itself).

For fixed ε define recursively the sets $\Omega_\varepsilon^{[n]}$ as follows. Let us call

$$\Omega_\varepsilon^{[0]} = \mathscr{D}_\tau(C_0), \tag{2.4}$$

and, for $n \geqslant 1$,

$$\Omega_\varepsilon^{[n]} = \left\{ \boldsymbol{\omega} \in \Omega_\varepsilon^{[n-1]} : |i\boldsymbol{\omega}\cdot\mathbf{v} - \mathscr{M}^{[n-1]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)| > \frac{C_0}{2^{(n+1)/2} |\nu|^\tau} \right\}; \tag{2.5}$$

finally define

$$\Omega_\varepsilon = \bigcap_{n=0}^{\infty} \Omega_\varepsilon^{[n]} = \lim_{n\to\infty} \Omega_\varepsilon^{[n]}, \tag{2.6}$$

and for any open subset $V \subset \mathbb{R}^d$ write $\Omega_\varepsilon^{[n]}(V) = \Omega_\varepsilon^{[n]} \cap V$.

Then Theorem 1 is a consequence of the following lemmata.

Lemma 1. Assume that the set Ω_ε has nonzero measure and that for all $\boldsymbol{\omega} \in \Omega_\varepsilon$ the functions $\mathscr{M}^{[p]}(\boldsymbol{\omega}, \mathbf{x}; \varepsilon)$ are C^1 in $\mathbf{x}$ and satisfy the bounds

$$|\mathscr{M}^{[p]}(\boldsymbol{\omega}, \mathbf{x}; \varepsilon)| \leqslant D\,|\varepsilon|, \qquad |\partial_{\mathbf{x}} \mathscr{M}^{[p]}(\boldsymbol{\omega}, \mathbf{x}; \varepsilon)| \leqslant D\,|\varepsilon|, \tag{2.7}$$

for some constant D and for all $p < n$. Then for any renormalized tree θ such that $\mathrm{Val}(\theta) \neq 0$ the number $N_n(\theta)$ of lines on scale n satisfies the bound

$$N_n(\theta) \leqslant c\, 2^{-n/(2\tau_1)} \sum_{v \in B(\theta)} |\mathbf{v}_v|, \tag{2.8}$$

for a suitable positive constant c.

Proof. One proceeds as in the proof of Lemma 1 of ref. 8, by proving inductively on the order k of the renormalized trees the bound

$$N_n^*(\theta) \leqslant \max\{0, 2\,|\mathbf{v}(\theta)|\, 2^{(3-n)/(2\tau_1)} - 1\}, \tag{2.9}$$

where $|\mathbf{v}(\theta)| \equiv \sum_{v \in B(\theta)} |\mathbf{v}_v|$ and $N_n^*(\theta)$ is the number of lines in $L(\theta)$ on scale $n' \geqslant n$. The changes with respect to ref. 8 are obvious, so that we omit details. ▮

Lemma 2. For $\boldsymbol{\omega} \in \Omega_\varepsilon$ and for $\mathbf{x}$ such that $g^{[n]}(\boldsymbol{\omega}, \mathbf{x}; \varepsilon) \neq 0$, there exist two constants D and D' such that the functions $\mathscr{M}^{[j]}(\boldsymbol{\omega}, \mathbf{x}; \varepsilon)$ are smooth functions of $\mathbf{x}$ and satisfy the bounds

$$
\begin{gathered}
|\mathscr{M}^{[j]}(\boldsymbol{\omega}, \mathbf{x}; \varepsilon)| \leqslant D\, |\varepsilon|, \qquad |\partial_{\mathbf{x}} \mathscr{M}^{[j]}(\boldsymbol{\omega}, \mathbf{x}; \varepsilon)| \leqslant D\, |\varepsilon|, \\
|\mathscr{M}^{[j]}(\boldsymbol{\omega}, \mathbf{x}; \varepsilon) - \mathscr{M}^{[j-1]}(\boldsymbol{\omega}, \mathbf{x}; \varepsilon)| \leqslant D\, |\varepsilon|\, e^{-D' 2^{j/\tau_1}},
\end{gathered}
\tag{2.10}
$$

for all $0 \leqslant j \leqslant n-1$.

Proof. The proof is by induction on j: again it can be carried out as in the proof of Lemma 2 in ref. 8, and it yields the following steps. First one proves that for all $T \in \mathscr{S}_{k,j}^{\mathscr{R}}$ contributing to $M^{[j]}(\boldsymbol{\omega}, \mathbf{x}; \varepsilon)$ through the self-energy value

$$
\mathscr{V}_T(\boldsymbol{\omega}, \mathbf{x}; \varepsilon) = \Bigg(\prod_{v \in E(T) \cup V(T)} F_v \Bigg) \Bigg(\prod_{\ell \in L(T)} g^{[n_\ell]}(\boldsymbol{\omega}, \mathbf{v}_\ell; \varepsilon) \Bigg) \tag{2.11}
$$

with $\mathbf{v}_\ell = \mathbf{v}_\ell^0 + \mathbf{x}$ (and $\mathbf{v}_\ell^0$ given by (7.15) in ref. 8), one must have

$$
\sum_{v \in B(T)} |\mathbf{v}_v| > 2^{(j-4)/(2\tau_1)}. \tag{2.12}
$$

Hence one proves that, by denoting with $N_{j'}(T)$ the number of lines on scale j' contained in $T \in \mathscr{S}_{k,j}^{\mathscr{R}}$, one has

$$
N_{j'}(T) \leqslant c\, 2^{-j'/(2\tau_1)} \sum_{v \in B(T)} |\mathbf{v}_v|; \tag{2.13}
$$

the proof makes use of an inductive application of Lemma 1 (see ref. 8 for further details). Then the two inequalities (2.12) and (2.13) imply, for all $T \in \mathscr{S}_{k,j}^{\mathscr{R}}$,

$$
|\mathscr{V}_T(\boldsymbol{\omega}, \mathbf{x}; \varepsilon)| \leqslant |\varepsilon|^k A_1 A_2^k e^{-A_3 2^{j/(2\tau_1)}} \prod_{v \in B(T)} e^{-\kappa |\mathbf{v}_v|/2}, \tag{2.14}
$$

for suitable constants A_1, A_2, and A_3.

By inserting the bound (2.13) into the definition of $M^{[j]}(\omega; \mathbf{x}; \varepsilon)$ we obtain

$$\begin{aligned}
|\mathcal{M}^{[j]}(\omega, \mathbf{x}; \varepsilon) - \mathcal{M}^{[j-1]}(\omega, \mathbf{x}; \varepsilon)| \\
&\leqslant |M^{[j]}(\omega, \mathbf{x}; \varepsilon)| \leqslant \sum_{k=1}^{\infty} D_1 D_2^k \, |\varepsilon|^k \, e^{-D_3 2^{j/(2\tau_1)}}, \\
|\mathcal{M}^{[j]}(\omega, \mathbf{x}; \varepsilon)| \leqslant \sum_{i=0}^{j} |M^{[i]}(\omega, \mathbf{x}; \varepsilon)| \\
&\leqslant \sum_{k=1}^{\infty} D_1 D_2^k \, |\varepsilon|^k \sum_{i=0}^{j} e^{-D_3 2^{i/(2\tau_1)}} \leqslant \sum_{k=1}^{\infty} \tilde{D}_1 D_2^k \, |\varepsilon|^k,
\end{aligned} \tag{2.15}$$

for suitable constants D_1, $\tilde{D}_1$, D_2, and D_3; this proves the first and third bounds in (2.7).

Also the second bound in (2.7) can be proved as in ref. 8. ∎

Lemma 3. The functions $\mathcal{M}^{[n]}(\omega, \mathbf{v}; \varepsilon)$ are C^1-extendible in the sense of Whitney outside $\Omega_\varepsilon^{[n-1]}$, and for all $\omega, \omega' \in \Omega_\varepsilon^{[n-1]}$ one has

$$\mathcal{M}^{[n]}(\omega', \mathbf{v}; \varepsilon) - \mathcal{M}^{[n]}(\omega, \mathbf{v}; \varepsilon) = (\omega' - \omega)\, \partial_\omega \mathcal{M}^{[n]}(\omega, \mathbf{v}; \varepsilon) + o(|\varepsilon|^2 \, |\omega' - \omega| \, |\mathbf{v}|), \tag{2.16}$$

where $\partial_\omega \mathcal{M}^{[n]}(\omega, \mathbf{v}; \varepsilon)$ is the formal derivative with respect to ω of $\mathcal{M}^{[n]}(\omega, \mathbf{v}; \varepsilon)$, and it admits the bound

$$|\partial_\omega \mathcal{M}^{[n]}(\omega, \mathbf{v}; \varepsilon)| \leqslant C \, |\varepsilon|^2 \, |\mathbf{v}|, \tag{2.17}$$

for some positive constant C.

Proof. The proof is by induction on n, and it can be performed by proceeding as in ref. 8, with the only difference that now ω plays the role of ε. For all $p \geqslant 0$ one has (see the last of (6.3) in ref. 8)

$$\begin{aligned}
M^{[p]}(\omega', \mathbf{v}; \varepsilon) - M^{[p]}(\omega, \mathbf{v}; \varepsilon) \\
= \sum_{k=1}^{\infty} \sum_{T \in \mathcal{S}_{k,p}^{\mathcal{R}}} \varepsilon^k \left(\prod_{v \in E(T) \cup V(T)} F_v \right) \\
\times \left[\left(\prod_{\ell \in L(T)} g^{[n_\ell]}(\omega', \mathbf{v}_\ell; \varepsilon) \right) - \left(\prod_{\ell \in L(T)} g^{[n_\ell]}(\omega, \mathbf{v}_\ell; \varepsilon) \right) \right].
\end{aligned} \tag{2.18}$$

Let us call $\Lambda(T)$ the set of lines in $L(T)$ coming out from nodes in $B(T)$. We can order the $|B(T)| - 1$ lines in $\Lambda(T)$ and construct a set of $|B(T)|$

subsets $\Lambda_1(T),\dots,\Lambda_{|B(T)|}(T)$ of $\Lambda(T)$, with $|\Lambda_j(T)|=j$, in the following way. Set $\Lambda_1(T)=\emptyset$, $\Lambda_2(T)=\ell_1$, if ℓ_1 is any line of T connected to the line coming out from T, and, if $|B(T)|\geqslant 3$, inductively for $2\leqslant j\leqslant |B(T)|-1$, $\Lambda_{j+1}(T)=\Lambda_j(T)\cup\ell_j$, where the line $\ell_j\in\Lambda(T)\setminus\Lambda_j(T)$ is connected to $\Lambda_j(T)$. Then in (2.18) we have

$$
\begin{aligned}
&\left(\prod_{\ell\in\Lambda(T)} g^{[n_\ell]}(\boldsymbol{\omega}',\mathbf{v}_\ell;\varepsilon)\right)-\left(\prod_{\ell\in\Lambda(T)} g^{[n_\ell]}(\boldsymbol{\omega},\mathbf{v}_\ell;\varepsilon)\right)\\
&\quad=\sum_{j=1}^{|B(T)|}\left[\left(\prod_{\ell\in\Lambda_j(T)} g^{[n_\ell]}(\boldsymbol{\omega}',\mathbf{v}_\ell;\varepsilon)\right)\left(g^{[n_{\ell_j}]}(\boldsymbol{\omega}',\mathbf{v}_{\ell_j};\varepsilon)-g^{[n_{\ell_j}]}(\boldsymbol{\omega},\mathbf{v}_{\ell_j};\varepsilon)\right)\right.\\
&\qquad\left.\times\left(\prod_{\ell\in\Lambda(T)\setminus(\Lambda_j(T)\cup\ell_j)} g^{[n_\ell]}(\boldsymbol{\omega},\mathbf{v}_\ell;\varepsilon)\right)\right],
\end{aligned}
\tag{2.19}
$$

where, by defining

$$
\begin{aligned}
&\Xi_n(\boldsymbol{\omega},\mathbf{v};\varepsilon)\\
&\quad=\chi_0(|\boldsymbol{\omega}\cdot\mathbf{v}|)\cdots\chi_{n-1}(|i\,\boldsymbol{\omega}\cdot\mathbf{v}-\mathscr{M}^{[n-2]}(\boldsymbol{\omega},\mathbf{v};\varepsilon)|)\;\chi_n(|i\,\boldsymbol{\omega}\cdot\mathbf{v}-\mathscr{M}^{[n-1]}(\boldsymbol{\omega},\mathbf{v};\varepsilon)|),\\
&\Psi_n(\boldsymbol{\omega},\mathbf{v};\varepsilon)\\
&\quad=\chi_0(|\boldsymbol{\omega}\cdot\mathbf{v}|)\cdots\chi_{n-1}(|i\,\boldsymbol{\omega}\cdot\mathbf{v}-\mathscr{M}^{[n-2]}(\boldsymbol{\omega},\mathbf{v};\varepsilon)|)\;\psi_n(|i\,\boldsymbol{\omega}\cdot\mathbf{v}-\mathscr{M}^{[n-1]}(\boldsymbol{\omega},\mathbf{v};\varepsilon)|),\\
&\Xi_{n,s}(\boldsymbol{\omega}',\boldsymbol{\omega},\mathbf{v};\varepsilon)\\
&\quad=\chi_0(|\boldsymbol{\omega}'\cdot\mathbf{v}|)\cdots\chi_{s-1}(|i\,\boldsymbol{\omega}'\cdot\mathbf{v}-\mathscr{M}^{[s-2]}(\boldsymbol{\omega}',\mathbf{v};\varepsilon)|)\\
&\qquad\times(\chi_s(|i\,\boldsymbol{\omega}'\cdot\mathbf{v}-\mathscr{M}^{[s-1]}(\boldsymbol{\omega}',v;\varepsilon)|)-\chi_s(|i\,\boldsymbol{\omega}\cdot\mathbf{v}-\mathscr{M}^{[s-1]}(\boldsymbol{\omega},\mathbf{v};\varepsilon)|))\\
&\qquad\times\chi_{s+1}(|i\,\boldsymbol{\omega}\cdot\mathbf{v}-\mathscr{M}^{[s]}(\boldsymbol{\omega},\mathbf{v};\varepsilon)|)\cdots\chi_n(|i\,\boldsymbol{\omega}\cdot\mathbf{v}-\mathscr{M}^{[n-1]}(\boldsymbol{\omega},\mathbf{v};\varepsilon)|),\qquad s\leqslant n,\\
&\Psi_{n,s}(\boldsymbol{\omega}',\boldsymbol{\omega},\mathbf{v};\varepsilon)\\
&\quad=\chi_0(|\boldsymbol{\omega}'\cdot\mathbf{v}|)\cdots\chi_{s-1}(|i\,\boldsymbol{\omega}'\cdot\mathbf{v}-\mathscr{M}^{[s-2]}(\boldsymbol{\omega}',\mathbf{v};\varepsilon)|)\\
&\qquad\times(\chi_s(|i\,\boldsymbol{\omega}'\cdot\mathbf{v}-\mathscr{M}^{[s-1]}(\boldsymbol{\omega}',v;\varepsilon)|)-\chi_s(|i\,\boldsymbol{\omega}\cdot\mathbf{v}-\mathscr{M}^{[s-1]}(\boldsymbol{\omega},\mathbf{v};\varepsilon)|))\\
&\qquad\times\chi_{s+1}(|i\,\boldsymbol{\omega}\cdot\mathbf{v}-\mathscr{M}^{[s]}(\boldsymbol{\omega},\mathbf{v};\varepsilon)|)\cdots\psi_n(|i\,\boldsymbol{\omega}\cdot\mathbf{v}-\mathscr{M}^{[n-1]}(\boldsymbol{\omega},\mathbf{v};\varepsilon)|),\qquad s<n,\\
&\Psi_{n,n}(\boldsymbol{\omega}',\boldsymbol{\omega},\mathbf{v};\varepsilon)\\
&\quad=\chi_0(|\boldsymbol{\omega}'\cdot\mathbf{v}|)\cdots\chi_{n-1}(|i\,\boldsymbol{\omega}'\cdot\mathbf{v}-\mathscr{M}^{[n-2]}(\boldsymbol{\omega}',\mathbf{v};\varepsilon)|)\\
&\qquad\times(\psi_n(|i\,\boldsymbol{\omega}'\cdot\mathbf{v}-\mathscr{M}^{[n-1]}(\boldsymbol{\omega}',v;\varepsilon)|)-\psi_n(|i\,\boldsymbol{\omega}\cdot\mathbf{v}-\mathscr{M}^{[n-1]}(\boldsymbol{\omega},\mathbf{v};\varepsilon)|)),
\end{aligned}
\tag{2.20}
$$

and setting $n_j = n_{\ell_j}$ and $\mathbf{v}_j = \mathbf{v}_{\ell_j}$, we obtain

$$\begin{aligned} g^{[n_j]}(\boldsymbol{\omega}', \mathbf{v}_j; \varepsilon) - g^{[n_j]}(\boldsymbol{\omega}, \mathbf{v}_j; \varepsilon) \\ = -\frac{[i(\boldsymbol{\omega}'-\boldsymbol{\omega})\cdot\mathbf{v} - (\mathscr{M}^{[n_j-1]}(\boldsymbol{\omega}', \mathbf{v}_j; \varepsilon) - \mathscr{M}^{[n_j-1]}(\boldsymbol{\omega}, \mathbf{v}_j; \varepsilon))]}{(i\boldsymbol{\omega}'\cdot\mathbf{v}_j - \mathscr{M}^{[n_j-1]}(\boldsymbol{\omega}', \mathbf{v}_j; \varepsilon))(i\boldsymbol{\omega}\cdot\mathbf{v}_j - \mathscr{M}^{[n_j-1]}(\boldsymbol{\omega}, \mathbf{v}_j; \varepsilon))} \Psi_{n_j}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon) \\ + \sum_{s=1}^{n_j} \frac{\Psi_{n_j, s}(\boldsymbol{\omega}', \boldsymbol{\omega}, \mathbf{v}; \varepsilon)}{i\boldsymbol{\omega}'\cdot\mathbf{v}_j - \mathscr{M}^{[n_j-1]}(\boldsymbol{\omega}', \mathbf{v}_j; \varepsilon')}. \end{aligned} \tag{2.21}$$

In $\Xi_{n,s}(\boldsymbol{\omega}', \boldsymbol{\omega}, \mathbf{v}; \varepsilon)$ and in $\Psi_{n,s}(\boldsymbol{\omega}', \boldsymbol{\omega}, \mathbf{v}; \varepsilon)$, with $s \leqslant n$, we can write (with the obvious interpretation for $\Psi_{n,n}(\boldsymbol{\omega}', \boldsymbol{\omega}, \mathbf{v}; \varepsilon)$)

$$\begin{aligned} \chi_s(|i\boldsymbol{\omega}'\cdot\mathbf{v} - \mathscr{M}^{[s-1]}(\boldsymbol{\omega}', v; \varepsilon)|) - \chi_s(|i\boldsymbol{\omega}\cdot\mathbf{v} - \mathscr{M}^{[s-1]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)|) \\ = \partial\chi_s(|i\boldsymbol{\omega}\cdot\mathbf{v} - \mathscr{M}^{[s-1]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)) \\ \times (|i\boldsymbol{\omega}'\cdot\mathbf{v} - \mathscr{M}^{[s-1]}(\boldsymbol{\omega}', \mathbf{v}; \varepsilon)| - |i\boldsymbol{\omega}\cdot\mathbf{v} - \mathscr{M}^{[s-1]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)|), \end{aligned} \tag{2.22}$$

where ∂ denotes the derivative with respect to the argument (so that one has $|\partial\chi_s(x)| \leqslant 2^s C_0^{-1} \Xi$, for some positive constant Ξ), and

$$\begin{aligned} \big||i\boldsymbol{\omega}'\cdot\mathbf{v} - \mathscr{M}^{[s-1]}(\boldsymbol{\omega}', \mathbf{v}; \varepsilon)| - |i\boldsymbol{\omega}\cdot\mathbf{v} - \mathscr{M}^{[s-1]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)|\big| \\ \leqslant |i(\boldsymbol{\omega}'-\boldsymbol{\omega})\cdot\mathbf{v} - (\mathscr{M}^{[s-1]}(\boldsymbol{\omega}', \mathbf{v}; \varepsilon) - \mathscr{M}^{[s-1]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon))| \\ \leqslant |i(\boldsymbol{\omega}'-\boldsymbol{\omega})\cdot\mathbf{v} - \partial_{\boldsymbol{\omega}}\mathscr{M}^{[s-1]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)(\boldsymbol{\omega}'-\boldsymbol{\omega})| + o(|\varepsilon|^2\, |\boldsymbol{\omega}'-\boldsymbol{\omega}|\, |\mathbf{v}|) \\ \leqslant (1 + C\, |\varepsilon|^2)\, |\boldsymbol{\omega}'-\boldsymbol{\omega}|\, |\mathbf{v}| + o(|\varepsilon|^2\, |\boldsymbol{\omega}'-\boldsymbol{\omega}|\, |\mathbf{v}|), \end{aligned} \tag{2.23}$$

where the inductive hypotheses (2.16) and (2.17), with n replaced by $p < n$, and the inclusion relations $\Omega_\varepsilon^{[n]} \subset \Omega_\varepsilon^{[p]}$, for $p < n$, have been used. Therefore we can write

$$\begin{aligned} \chi_s(|i\boldsymbol{\omega}'\cdot\mathbf{v} - \mathscr{M}^{[s-1]}(\boldsymbol{\omega}', v; \varepsilon)|) - \chi_s(|i\boldsymbol{\omega}\cdot\mathbf{v} - \mathscr{M}^{[s-1]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)|) \\ = \mathbf{K}_s(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)(\boldsymbol{\omega}'-\boldsymbol{\omega}) + o(2^s C_0^{-1}\, |\boldsymbol{\omega}'-\boldsymbol{\omega}|\, |\mathbf{v}|), \end{aligned} \tag{2.24}$$

with the function $\mathbf{K}_s(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)$ admitting the bound $|\mathbf{K}_s(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)| \leqslant 2^s C_0^{-1} K\, |\mathbf{v}|$, for a suitable positive constant K.

We can also write in (2.21), for $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \Omega_\varepsilon^{[n]}$ and for $n_j \leqslant n$,

$$\begin{aligned} g^{[n_j]}(\boldsymbol{\omega}', \mathbf{v}_j; \varepsilon) - g^{[n_j]}(\boldsymbol{\omega}, \mathbf{v}_j; \varepsilon) \\ = \partial_{\boldsymbol{\omega}} g^{[n_j]}(\boldsymbol{\omega}, \mathbf{v}_j; \varepsilon)(\boldsymbol{\omega}'-\boldsymbol{\omega}) + o(2^{2n_j} C_0^{-2}\, |\boldsymbol{\omega}'-\boldsymbol{\omega}|\, |\mathbf{v}_j|), \end{aligned} \tag{2.25}$$

where $\partial_{\omega} g^{[n_j]}(\omega, \mathbf{v}_j; \varepsilon)$ represents the formal derivative of $g^{[n_j]}(\omega, \mathbf{v}_j; \varepsilon)$ with respect to ω, and it admits the bound

$$|\partial_{\omega} g^{[n_j]}(\omega, \mathbf{v}_j; \varepsilon)| \leqslant G_1 2^{2n_j} C_0^{-2} |\mathbf{v}_j|, \tag{2.26}$$

for some constant G_1.

If $\Psi_{n_j}(\omega', \mathbf{v}_j; \varepsilon) \neq 0$ this follows from (2.21) and from the inductive hypothesis, by using (2.16), with $p < n$ instead of n, and (2.24) in order to obtain the bound

$$\begin{aligned} |\partial_{\omega} g^{[n_j]}(\omega \cdot \mathbf{v}_j; \varepsilon)| &\leqslant \frac{1}{C_0^2 2^{-2(n_j+1)}} (1 + C |\varepsilon|^2) |\mathbf{v}| + \frac{1}{C_0 2^{-2(n_j+1)}} \sum_{s=0}^{n_j} 2^s C_0^{-1} K |\mathbf{v}| \\ &\leqslant D 2^{2(n_j+1)} C_0^{-2} |\mathbf{v}|, \end{aligned} \tag{2.27}$$

where D is a suitable positive constant.

If $\Psi_{n_j}(\omega', \mathbf{v}_j; \varepsilon) = 0$ then we have

$$g^{[n_j]}(\omega', \mathbf{v}_j; \varepsilon) - g^{[n_j]}(\omega, \mathbf{v}_j; \varepsilon) = -g^{[n_j]}(\omega, \mathbf{v}_j; \varepsilon). \tag{2.28}$$

By defining $D_n(\omega, \mathbf{v}; \varepsilon) = i\omega \cdot \mathbf{v} - \mathscr{M}^{[n-1]}(\omega, \mathbf{v}; \varepsilon)$, for $D_{n_j}(\omega', \mathbf{v}_j; \varepsilon) \geqslant D_{n_j}(\omega, \mathbf{v}_j; \varepsilon)$ the relation (2.25) and the bound (2.26) easily follow again from (2.21). For $D_{n_j}(\omega', \mathbf{v}_j; \varepsilon) < D_{n_j}(\omega, \mathbf{v}_j; \varepsilon)$, we can write, by the inductive hypothesis,

$$\begin{aligned} D_{n_j}(\omega', \mathbf{v}_j; \varepsilon) &= D_{n_j}(\omega, \mathbf{v}_j; \varepsilon) \\ &\quad + (i\mathbf{v} - \partial_{\omega} \mathscr{M}^{[n_j-1]}(\omega, \mathbf{v}_j; \varepsilon))(\omega' - \omega) + o(|\varepsilon|^2 |\omega' - \omega| |\mathbf{v}_j|). \end{aligned} \tag{2.29}$$

Then, if $|\omega' - \omega| |\mathbf{v}_j| \leqslant |D_{n_j}(\omega, \mathbf{v}_j; \varepsilon)|/4$, we can bound $D_{n_j}^{-1}(\omega', \mathbf{v}_j; \varepsilon)$ with $4D_{n_j}^{-1}(\omega, \mathbf{v}_j; \varepsilon)$, so that (2.25) and (2.26) follow immediately, while, if $|\omega' - \omega| |\mathbf{v}_j| > |D_{n_j}(\omega, \mathbf{v}_j; \varepsilon)|/4$, we can bound in (2.28)

$$\begin{aligned} |g^{[n_j]}(\omega, \mathbf{v}_j; \varepsilon)| &\leqslant |D_{n_j}^{-1}(\omega, \mathbf{v}_j; \varepsilon)| \\ &\leqslant 4 |D_{n_j}^{-2}(\omega, \mathbf{v}_j; \varepsilon)| |\omega' - \omega| |\mathbf{v}_j| \\ &\leqslant 4\, 2^{2(n_j+1)} C_0^{-2} |\omega' - \omega| |\mathbf{v}_j|, \end{aligned} \tag{2.30}$$

hence (2.25) and (2.26) follow also in such a case.

To obtain (2.16) we have to consider

$$\begin{aligned}
&\mathscr{M}^{[n]}(\boldsymbol{\omega}', \mathbf{v}; \varepsilon) - \mathscr{M}^{[n]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon) \\
&\qquad = \sum_{p=0}^{n} \Xi_p(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)(M^{[p]}(\boldsymbol{\omega}', \mathbf{v}; \varepsilon) - M^{[p]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)) \\
&\qquad\quad + \sum_{p=0}^{n} \sum_{s=0}^{p} \Xi_{p,s}(\boldsymbol{\omega}', \boldsymbol{\omega}, \mathbf{v}; \varepsilon)\, M^{[p]}(\boldsymbol{\omega}', \mathbf{v}; \varepsilon).
\end{aligned} \tag{2.31}$$

We can insert (2.18) into (2.31) and express the last line of (2.18) through (2.19), hence the difference of propagators (2.19) through (2.25), and we find that (2.16) holds, provided it is true for $n = 0$.

The validity of (2.16)—and (2.17)—for $n = 0$ follows by explicitly calculation by noting that in such a case we can still use the decompositions (2.25) and (2.18), with the only difference that now all propagators are of the form $g^{[0]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon) = \psi_0(|\boldsymbol{\omega} \cdot \mathbf{v}|)(i\boldsymbol{\omega} \cdot \mathbf{v})^{-1}$. Hence we have to consider

$$\begin{aligned}
&\mathscr{M}^{[0]}(\boldsymbol{\omega}', \mathbf{v}; \varepsilon) - \mathscr{M}^{[0]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon) \\
&\quad = \chi_0(|\boldsymbol{\omega} \cdot \mathbf{v}|) \Bigg\{ \sum_{k=1}^{\infty} \sum_{T \in \mathscr{S}^{\mathscr{R}}_{k,0}} \varepsilon^k \Bigg(\prod_{v \in E(T) \cup V(T)} F_v \Bigg) \\
&\quad\quad \times \sum_{j=1}^{|B(T)|} \Bigg[\Bigg(\prod_{\ell \in \Lambda_j(T)} g^{[0]}(\boldsymbol{\omega}', \mathbf{v}_\ell; \varepsilon) \Bigg) (g^{[0]}(\boldsymbol{\omega}', \mathbf{v}_{\ell_j}; \varepsilon) - g^{[0]}(\boldsymbol{\omega}, \mathbf{v}_{\ell_j}; \varepsilon)) \\
&\quad\quad \times \Bigg(\prod_{\ell \in \Lambda(T) \setminus (\Lambda_j(T) \cup \ell_j)} g^{[0]}(\boldsymbol{\omega}, \mathbf{v}_\ell; \varepsilon) \Bigg) \Bigg] \Bigg\} \\
&\quad\quad + (\chi_0(|\boldsymbol{\omega}' \cdot \mathbf{v}|) - \chi_0(|\boldsymbol{\omega} \cdot \mathbf{v}|))\, M^{[0]}(\boldsymbol{\omega}', \mathbf{v}; \varepsilon),
\end{aligned} \tag{2.32}$$

with

$$\begin{aligned}
&g^{[0]}(\boldsymbol{\omega}', \mathbf{v}_{\ell_j}; \varepsilon) - g^{[0]}(\boldsymbol{\omega}, \mathbf{v}_{\ell_j}; \varepsilon) \\
&\quad = \frac{\psi_0(|\boldsymbol{\omega}' \cdot \mathbf{v}|)}{i\boldsymbol{\omega}' \cdot \mathbf{v}_{\ell_j}} - \frac{\psi_0(|\boldsymbol{\omega} \cdot \mathbf{v}|)}{i\boldsymbol{\omega} \cdot \mathbf{v}_{\ell_j}} \\
&\quad = -\frac{(\boldsymbol{\omega}' - \boldsymbol{\omega}) \cdot \mathbf{v}_{\ell_j}}{(i\boldsymbol{\omega}' \cdot \mathbf{v}_{\ell_j})(i\boldsymbol{\omega} \cdot \mathbf{v}_{\ell_j})} \psi_0(|\boldsymbol{\omega} \cdot \mathbf{v}|) + \frac{\psi_0(|\boldsymbol{\omega}' \cdot \mathbf{v}|) - \psi_0(|\boldsymbol{\omega} \cdot \mathbf{v}|)}{i\boldsymbol{\omega}' \cdot \mathbf{v}_{\ell_j}},
\end{aligned} \tag{2.33}$$

which is C^1 extendible outside $\Omega^{[0]}_{\varepsilon}$. Then the formal derivative of $\mathscr{M}^{[0]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)$ with respect to $\boldsymbol{\omega}$ produces a sum of terms which are of the form of some quantity which can be bounded by the square of the bound holding for $\mathscr{M}^{[n]}(\boldsymbol{\omega} \cdot \mathbf{v}; \varepsilon)$ (simply because, as we have seen by reasoning as in proving (2.25) and (2.26), the difference of propagators (2.33) in (2.32) can be bounded by a constant times the product of the bounds of the single propagators) times $|\mathbf{v}_\ell|$, where ℓ is some line in $\Lambda(T)$. So we can write $\mathbf{v}_\ell = \mathbf{v}_\ell^0 + \sigma_\ell \mathbf{v}$, with $\sigma_\ell \in \{0, 1\}$: the term with $\mathbf{v}_\ell^0$ can be easily estimated since the product of node factors in the self-energy value can be bounded by a quantity containing an exponentially decaying factor $e^{-\kappa |\mathbf{v}_{\ell_0}|}$ (because of the analyticity of $f(t)$), while the other one (when not vanishing) is just of the form of a constant times $|\varepsilon|^2$ times $|\mathbf{v}|$.

To show that the bound (2.17) holds also for $n \geqslant 1$, we can reason essentially in the same way, again by using the bounds (2.22) and (2.25) which follow from the inductive hypothesis.

This concludes the proof. ▮

Lemma 4. For any ball $V \subset \mathbb{R}^d$ of Lebesgue measure 1, there is a positive constant b_V such that, for ε_0 small enough anf for $|\varepsilon| < \varepsilon_0$, one has

$$\operatorname{meas}(\Omega_\varepsilon(V)) \geqslant 1 - b_V C_0, \tag{2.34}$$

where meas denotes the Lebesgue measure.

Proof. Fix any ball $V \subset \mathbb{R}^d$. Define

$$\begin{cases} \mathscr{I}^{[0]} = \emptyset, \\ \mathscr{I}^{[n]} = \Omega_\varepsilon^{[n-1]}(V) \setminus \Omega_\varepsilon^{[n]}(V), \qquad \text{for} \quad n \geqslant 1; \end{cases} \tag{2.35}$$

note that $\mathscr{I} \equiv \bigcup_{n=0}^{\infty} \mathscr{I}^{[n]} = \mathscr{D}_\tau(C_0) \cap V \setminus \Omega_\varepsilon(V)$.

For all $n \geqslant 1$ and for all $\mathbf{v} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ define

$$I^{[n]}(\mathbf{v}) = \left\{ \boldsymbol{\omega} \in \Omega_\varepsilon^{[n-1]}(V) : |i \boldsymbol{\omega} \cdot \mathbf{v} - \mathscr{M}^{[n-1]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)| \leqslant \frac{C_0}{2^{(n+1)/2} \, |\mathbf{v}|^\tau} \right\}. \tag{2.36}$$

For any $\mathbf{v}$ set $\boldsymbol{\omega} = \alpha \mathbf{v}/|\mathbf{v}| + \boldsymbol{\beta}$, with $\boldsymbol{\beta} \cdot \mathbf{v} = 0$: α is the component of $\boldsymbol{\omega}$ along the direction $\mathbf{v}$, while $\boldsymbol{\beta}$ is the orthogonal component, so that $\boldsymbol{\omega} \cdot \mathbf{v} = \alpha \, |\mathbf{v}|$.

Therefore one has to exclude from the set $\Omega_\varepsilon^{[n-1]}(V)$ all the values $\boldsymbol{\omega} = \alpha \mathbf{v}/|\mathbf{v}| + \boldsymbol{\beta}$ in $I^{[n]}(\mathbf{v})$, which gives a set of measure

$$\int_{I^{[n]}(\mathbf{v})} d\boldsymbol{\omega} = \int_{I^{[n]}(\mathbf{v})} d\boldsymbol{\beta} \, d\alpha = \int d\boldsymbol{\beta} \int_{-1}^{1} dt \, \frac{d\alpha(t)}{dt}, \tag{2.37}$$

if $\alpha(t)$ is defined by

$$i\alpha(t) \, |\mathbf{v}| - \mathscr{M}^{[n]}(\alpha(t) \, \mathbf{v}/|\mathbf{v}| + \boldsymbol{\beta}, \mathbf{v}; \varepsilon) = t \, \frac{C_0}{2^{(n+1)/2} \, |\mathbf{v}|^\tau}, \tag{2.38}$$

where the Whitney extension of $\mathscr{M}^{[n]}(\boldsymbol{\omega}, \mathbf{v}; \varepsilon)$ given by Lemma 3 has to be used.

By deriving (2.38) with respect to t we obtain

$$\frac{d\alpha(t)}{dt} |\mathbf{v}| \left(i - \frac{\mathbf{v}}{|\mathbf{v}|^2} \partial_{\boldsymbol{\omega}} \mathscr{M}^{[n]}(\alpha(t)\, \mathbf{v}/|\mathbf{v}| + \boldsymbol{\beta}, \mathbf{v}; \varepsilon) \right) = \frac{C_0}{2^{(n+1)/2}\, |\mathbf{v}|^{\tau}}, \tag{2.39}$$

so that, by using (2.17), we can bound

$$\left| \frac{d\alpha(t)}{dt} \right| \leqslant \frac{1}{1 - C\, |\varepsilon|^2} \frac{C_0}{2^{(n+1)/2}\, |\mathbf{v}|^{\tau+1}} \leqslant \frac{2C_0}{2^{(n+1)/2}\, |\mathbf{v}|^{\tau+1}}, \tag{2.40}$$

which, inserted into (2.33), gives

$$\int_{I^{[n]}(\mathbf{v})} d\boldsymbol{\omega} \leqslant C_V\, \frac{C_0}{2^{(n+1)/2}\, |\mathbf{v}|^{\tau+1}}, \tag{2.41}$$

for some constant C_V.

This has to be done for all $\mathbf{v} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, so that we have to exclude from $\Omega_{\varepsilon}^{[n-1]}(V)$ a set

$$\mathscr{I}^{[n]} = \bigcup_{\mathbf{v} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} I^{[n]}(\mathbf{v}) \tag{2.42}$$

of measure bounded by

$$\begin{aligned} \text{meas}(\mathscr{I}^{[n]}) &\leqslant \sum_{\mathbf{v} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \text{meas}(I^{[n]}(\mathbf{v})) \\ &\leqslant C_V \sum_{\mathbf{v} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \frac{C_0}{2^{(n+1)/2}\, |\mathbf{v}|^{\tau+1}} \leqslant B_V\, \frac{C_0}{2^{(n+1)/2}}, \end{aligned} \tag{2.43}$$

with B_V implicitly defined, as the sum is convergent for $\tau > d-1$.

We are left with the sum over n to perform, but this can be done trivially, and it gives

$$\text{meas}(\mathscr{I}) \leqslant \sum_{n=0}^{\infty} \text{meas}(\mathscr{I}^{[n]}) \leqslant B_V C_0 \sum_{n=0}^{\infty} \frac{1}{2^{(n+1)/\tau}} \leqslant b_V C_0, \tag{2.44}$$

for some positive constant b_V. ▮

Lemmata 1 and 2 prove that there exists $\varepsilon_0 > 0$ such that the series (2.1) converges for all $|\varepsilon| < \varepsilon_0$ provided that one has $\boldsymbol{\omega} \in \Omega_{\varepsilon}$. In fact by using Lemma 1 one proves that the functions $\mathscr{M}^{[n]}(\boldsymbol{\omega}, \mathbf{v}; \boldsymbol{\omega})$ admit the bounds stated in Lemma 2. This in turn implies that the bound (2.8) on the number of lines on scale n holds for all $n \geqslant 0$, so that one can reason as in ref. 8 to conclude that the coefficients $u_{\nu}^{[k]}$ admit the bound $C^k e^{-\kappa' |\nu|}$ for some positive constants C and κ': then convergence follows. Lemmata 3 and 4 prove that the set Ω_ε is not empty: more precisely for any ball $V \subset \mathbb{R}^d$ (of not too small Lebesgue measure) the intersection $\Omega_\varepsilon \cap V$ has large relative measure provided that the Diophantine constant C_0 is small enough.

3. RESULTS ABOUT THE SPECTRUM OF THE QUASI-ENERGY OPERATOR

If the spectrum of the quasi-energy operator (1.3) is pure point, then any solution of the Schrödinger equation (1.4) has to be a linear combination of harmonics

$$\exp(i(\boldsymbol{\omega} \cdot \boldsymbol{\nu} + \lambda_m)\, t), \qquad m = 1, 2, \quad \boldsymbol{\nu} \in \mathbb{Z}^2, \tag{3.1}$$

for suitable λ_1 and λ_2; see ref. 5.

To deduce (3.1) from the results of Section 2 requires some considerations, as any particular quasi-periodic solution $g(t)$ of the generalized Riccati equation (1.5) and the solution $\psi(t)$ of the Schrödinger equation (1.4) are related according to Theorem 2.1 of ref. 2; see also ref. 3. So one has

$$\psi(t) = e^{i\pi\sigma_2/4} U(t)\, e^{-i\pi\sigma_2/4} \psi(0), \tag{3.2}$$

with $\psi(0) \in \mathbb{C}^2$ and the unitary transformation $U(t)$ given by formula (2.3) of ref. 2:

$$U(t) = \begin{pmatrix} R(t)(1 + ig(0)\, S(t)) & -i\varepsilon R(t)\, S(t) \\ -i\varepsilon \overline{R(t)\, S(t)} & \overline{R(t)(1 - ig(0)\, S(t))} \end{pmatrix}, \tag{3.3}$$

where

$$R(t) = \exp\left(-i \int_0^t \mathrm{d}t' (f(t') + g(t')) \right), \qquad S(t) = \int_0^t \mathrm{d}t'\, R^{-2}(t'), \tag{3.4}$$

and the bar denotes complex conjugation as in (1.2). In particular, in order to have a quasi-periodic solution one needs not only that $g(t)$ is quasi-periodic, but also, as remarked in ref. 2, that it satisfies some properties, which would be difficult to prove directly from the perturbative expansion of $g(t)$, but which follow automatically from the unitarity of $U(t)$, namely one has

$$g_{\mathbf{0}} \in \mathbb{R}, \qquad (R^{-2})_{\mathbf{0}} = 0. \tag{3.5}$$

Even if such properties are satisfied, in general it is not obvious that $\psi(t)$ can be written as a linear combination of the functions (3.1). However there is a case in which this can be easily checked, and it is exactly the case of an external field with zero average which was left unsolved in literature (as it represents a degenerate, hence more complicated, situation in the direct approach of ref. 11), so we think that it can be of interest to discuss it explicitly.

If $f_0 = 0$ then the solution $g(t)$ of the generalized Riccati equation (1.5) is a quasi-periodic function depending on time through the quantity $\boldsymbol{\omega}t$ (with the notations in refs. 2 and 8 one has $d' = d$ and $\underline{\omega} = \boldsymbol{\omega}$), so that, if the conditions (3.5) are satisfied, it is straightforward to check that $\psi(t)$ can be written as a linear combination of the functions (3.1) with $\lambda_1 = -g_0$ and $\lambda_2 = g_0$: this proves that the spectrum of the quasi-energy operator is pure point, and it provides an explicit expansion for the eigenvalues.

ACKNOWLEDGMENTS

I'm indebted with João Barata and Giovanni Gallavotti for useful discussions.

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